Estimation of Base Station Position Using Timing Advance Measurements

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***Abstract*—Timing Advance is used in TDMA (Time Division Multiple Access) systems, such as GSM and LTE, to synchronize the mobile phone to the cellular BS (Base Station). Mobile phone positioning can use TA measurements if BS positions are known, but in many cases BS positions are not in the public domain. In this work we study how to use a set of TA measurements taken by mobile phones at known positions to estimate the position of a BS. This paper describes two methods -- GMF (Gaussian Mixture Filter) and PMF (Point Mass Filter) for estimation of the BS position. Positioning performance is evaluated using simulated and real measurements. In suburban field tests, TA measurements suffice to determine BS position with an error comparable to the TA granularity (550m). GMF computes BS position much faster than PMF and is only slightly less accurate.**

Keywords - Timing advance; Nonlinear filters; Mobile communication; Position measurement; Positioning; GSM;

I. INTRODUCTION

Mobile phone positioning may be done, for example, using GNSS (Global Navigation Satellite System) or using information of cellular network. Many mobile phones have an integrated GNSS receiver and in most cases GNSS position estimate is more accurate than positioning based on the network information. However, many low-end mobile phones do not have integrated GNSS receiver. Even for GNSS-equipped phones, cell based positioning has some advantages to GNSS. First position fix in GNSS may take a while or may be impossible due to signal blocking. Also the cellular positioning accuracy is good enough for some applications, for example, local weather information. Benefits of cellular methods in these cases compared to GNSS methods are the much lower power consumption, faster initial position fix and better availability.

When the serving cell's geographical coverage area is known, the position of a mobile phone may be estimated e.g. as the coverage area centroid. If the handset is connected to several networks or can hear several BSs (Base Stations) the position can be estimated by consideration of the overlapping coverage areas [1]. In addition to the cell identity other network parameters may be used in positioning e.g. received signal strength, TA (Timing Advance) in GSM (Global System for Mobile Communications), LTE (Long Term Evolution) and TD-SCDMA (Time Division Synchronous Code Division Multiple Access) networks, AoA (Angle of Arrival), TDOA (Time Difference of Arrival), among others. For a comprehensive study of these methods see [2].

If the handset has WLAN (Wireless Local Area Network) capabilities, the cell positioning may be done exploiting heard WLAN stations. WLAN BS has a significantly smaller range than GSM BS, and this leads to better positioning accuracy. Problems with WLAN cell positioning are the scarcity of WLANs in rural areas and the possibility that WLAN BSs may be mobile. Mapping of personal WLAN access points also raises privacy issues.

TA can be used to estimate the distance of mobile phone to BS and these range measurements can be used for mobile phone positioning, either standalone or combined with other measurements such as cell id [3]. TA-based mobile phone positioning can be appropriate in rural settings where cells are large and for some reason GNSS is not available. However, for the method to work, it is necessary to know the BS locations, and this information is not always in the public domain. For this reason, there is interest in methods to determine locations of BSs.

In this paper we show how BS position can be estimated from a set of TA measurements taken by GNSS-equipped mobile phones. In the next section the TA is presented. In Section III methods for BS position estimation are presented. Results are presented in Section IV and future work in Section V. Section VI concludes the article.

II. TIMING ADVANCE

TA is the RTT (Round Trip Time) from mobile phone to BS. It is specified in [4] and is used to minimize interference in TDMA (Time-Division Multiple Access) systems. When multiple mobile phones are sending on the same physical channel they need to know the right time to send so that data arrives to BS antenna on the right timeslot. The mobile phone and BS do initial synchronization on RACH (Random Access Channel) using zero timing advance. After this the BS tells the mobile phone how much transmission has to be advanced. Granularity of TA is a GSM bit (48/13μs). TA is not always available in mobile phone e.g. when the radio link is in idle-state TA is not available.

TA measurement may be transformed to a discrete distance measurement with granularity of $\Delta_{TA}=c\cdot 24/13\mu s\approx$ 550m. In LTE networks TA will have much finer granularity of 16/(15000·2048)s that corresponds to ≈78m in range [5, 6], but because the LTE networks and devices are not currently widespread we here will concentrate on GSM TA.

A. Ideal Model

The discrete TA measurement presented above may be also written as

$$\mathrm{TA} = \mathrm{floor}\left(\frac{\left\|\mathbf{x}^{\mathrm{M}} - \mathbf{x}^{\mathrm{BS}}\right\|}{\Delta_{\mathrm{TA}}}\right), \tag{1}$$

where $\mathbf{x}^{\mathrm{M}}$ is the position of the mobile phone and $\mathbf{x}^{\mathrm{BS}}$ is the BS position. In this paper we assume that $\mathbf{x}^{\mathrm{M}}$ is known (it is computed using mobile phone GNSS). Based on this model we can write the likelihood of the ideal TA measurement

$$p(\mathrm{TA} \mid \mathbf{x}^{\mathrm{BS}}) = \begin{cases} 1, & 0 \leq \left\|\mathbf{x}^{\mathrm{M}} - \mathbf{x}^{\mathrm{BS}}\right\| - \mathrm{TA}\Delta_{\mathrm{TA}} < \Delta_{\mathrm{TA}} \\ 0, & \text{otherwise} \end{cases}. \tag{2}$$

B. Measurement Model from Real Data

In practice the ideal model presented above does not hold because of errors. TA measurement may have different values at same distance from the BS or even on the same spot at different times. Some TA values measured in different locations and the position of the BS are shown in Fig. 1.

In this paper we assume that the real measurement model (likelihood) is rotationally symmetric. Using this assumption it is enough to approximate from the real data one-dimensional likelihood $p(\mathrm{TA}|r)$, where $r=\|\mathbf{x}^{\mathrm{M}}-\mathbf{x}^{\mathrm{BS}}\|$. Note that the ideal likelihood $p(\mathrm{TA}|r)$ (see (2)) is proportional to pdf (probability density function) of uniform distribution

$$\mathrm{Uniform}\left(\mathrm{TA}\Delta_{\mathrm{TA}}, (\mathrm{TA}+1)\Delta_{\mathrm{TA}}\right), \tag{3}$$

whose mean is $(2\mathrm{TA}\Delta_{\mathrm{TA}}+\Delta_{\mathrm{TA}})/2$ and variance is $(\Delta_{\mathrm{TA}})^2/12$.

One option to approximate likelihood is to use following algorithm. First the n measurements are sorted in ascending order by the distance to the BS. The weight of each measurement is then set to

$$w_i = \begin{cases} k_{\mathrm{TA}_i} d_2 & , i = 1 \\ k_{\mathrm{TA}_i} (d_{i+1} - d_{i-1}) & , 1 < i < n \\ 2k_{\mathrm{TA}_i} (d_i - d_{i-1}) & , i = n \end{cases}, \tag{4}$$

where k_{TA_i} are such that $\sum_{\mathrm{TA}_i = ta} w_i = 1$ for all received TA values (ta) and d_i is the distance from a measurement to the BS. Now we can approximate, that

$$p(\mathrm{TA} = ta \mid r) \propto \mathrm{N}_{\sigma_{ta}^2}^{\mu_{ta}}(r), \tag{5}$$

where $\mathrm{N}_{\sigma_{\mathrm{ta}}^2}^{\mu_{\mathrm{ta}}}(r)$ is pdf of the one-dimensional Gaussian distribution $\mathrm{N}(\mu, \sigma^2)$

$$\mu_{ta} = \sum_{\mathrm{TA}_i = ta} w_i d_i \text{ and} \tag{6}$$

$$\sigma_{ta}^2 = \sum_{\mathrm{TA}_i = ta} w_i (d_i - \mu_{ta})^2. \tag{7}$$

We consider two different Gaussian models for TA measurement. In the first model (5) the mean and variance are fitted for each separate TA measurement value independently. In the second model we subtract $ta\Delta_{\mathrm{TA}}$ from each distance and use just a single measurement model for all different TA values. In positioning the variance σ_{ta}^2 is the same for all TA measurements and the means are

$$\mu_{\mathrm{ta}} = \mu_0 + ta\Delta_{\mathrm{TA}}. \tag{8}$$

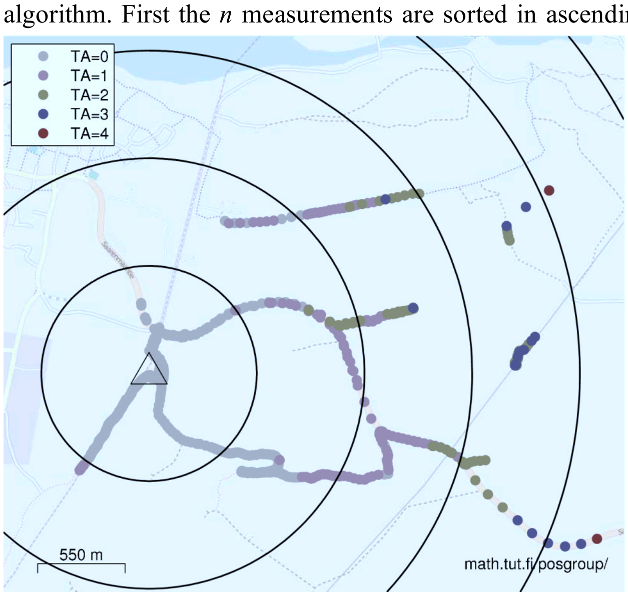

Figure 1. TA-measurements from one base station in different positions. The circles are based on the ideal model (2).

III. Base Station Position Estimation

In following sections two different methods for estimating BS position are presented. The first method is PMF (Point Mass Filter) [7, 8] that is a quite simple and converges to the exact posterior distribution as the number of computation points increases. The second method is GMF (Gaussian Mixture Filter) [9-12] that estimates the probability distribution using multiple Gaussians and is faster than PMF [13]. Both of these filters, PMF and GMF are Bayesian filters, whose aim is to compute posterior distribution of the BS position $p(\mathbf{x}^{\mathrm{BS}}|\mathrm{TA}_{1:n})$, given all available TA measurements $\mathrm{TA}_{1:n}=\{\mathrm{TA}_1,\mathrm{TA}_2,\ldots,\mathrm{TA}_n\}$. We assume:

- TA measurements are conditionally independent given the $\mathbf{x}^{\mathrm{BS}}$
- BS position is static
- The initial prior distribution $p\left(\mathbf{x}_{t_0}^{\mathrm{BS}} \mid \mathrm{TA}_{1:0}\right) = p\left(\mathbf{x}_{t_0}^{\mathrm{BS}}\right)$ is uniform
- Local 2D coordinate system is used
- First measurement is at the origin of the coordinate system
- Each measurement position has zero error (i.e. position error has been incorporated into the TA measurement model)

The posterior distribution can be computed recursively:

$$\text{Prediction: } p\left(\mathbf{x}_{t_k}^{\mathrm{BS}} \mid \mathrm{TA}_{1:k-1}\right) = p\left(\mathbf{x}_{t_{k-1}}^{\mathrm{BS}} \mid \mathrm{TA}_{1:k-1}\right) \quad (9)$$

$$\text{Update: } p\left(\mathbf{x}_{t_k}^{\mathrm{BS}} \mid \mathrm{TA}_{1:k}\right) \propto p\left(\mathrm{TA}_k \mid \mathbf{x}_{t_k}^{\mathrm{BS}}\right) p\left(\mathbf{x}_{t_k}^{\mathrm{BS}} \mid \mathrm{TA}_{1:k-1}\right) \quad (10)$$

A. Point Mass Filter

PMF approximates posterior distributions using convex combinations of delta functions (point masses)

$$p(\mathbf{x}^{\mathrm{BS}} \mid \mathrm{TA}_{1:k}) = \sum_{j=1}^{N_{\mathrm{PMF}}} \omega_j \delta(\mathbf{x}^{\mathrm{BS}} - \mathbf{x}_j^{\mathrm{PMF}}), \quad (11)$$

where points $\mathbf{x}_j^{\mathrm{PMF}}$ are deterministically set. Here $\omega_j \geq 0$ and $\sum_{j=1}^{N_{\mathrm{PMF}}} \omega_i = 1$. In this paper points are chosen on an equispaced grid.

B. Gaussian Mixture Filter

GMF approximates posterior distributions using convex combinations of Gaussian distributions:

$$p(\mathbf{x}^{\mathrm{BS}} \mid \mathrm{TA}_{1:k}) = \sum_{j=1}^{N_{\mathrm{GMF}}} \alpha_j \mathrm{N}_{\mathrm{P}_j}^{\mu_j}(\mathbf{x}^{\mathrm{BS}}), \quad (12)$$

where $\mathrm{N}_{\mathrm{P}_j}^{\mu_j}(\mathbf{x}^{\mathrm{BS}})$ is pdf of 2D Gaussian distribution $\mathrm{N}(\mu, \mathrm{P})$. Here $\alpha_j \geq 0$ and $\sum_{j=1}^{N_{\mathrm{GMF}}} \alpha_i = 1$. We use abbreviation

$$\mathrm{M}(\alpha_j, \mu_j, \mathrm{P}_j)_{(j, N_{\mathrm{GMF}})} \quad (13)$$

to present the full GMF. Algorithm of general GMF is given below.

Algorithm 1 Gaussian Mixture Filter

Initialization of GMF at time t_1 (Sec. III-B1):

$$\mathrm{M}(\alpha_{j,1}, \mu_{j,1}, \mathrm{P}_{j,1})_{(j, N_{\mathrm{GMF},1})}$$

for $k = 2$ to n **do**

1. *Prediction:*
$$\mathrm{M}(\alpha_{j,k}^-, \mu_{j,k}^-, \mathrm{P}_{j,k}^-)_{(j, N_{\mathrm{GMF}^-,k})} := \mathrm{M}(\alpha_{j,k-1}, \mu_{j,k-1}, \mathrm{P}_{j,k-1})_{(j, N_{\mathrm{GMF}^-,k-1})}$$
2. *Update:* Compute posterior approximation using a bank of EKFs (Sec. III-B2)
$$\mathrm{M}(\alpha_{j,k}^+, \mu_{j,k}^+, \mathrm{P}_{j,k}^+)_{(j, N_{\mathrm{GMF}^+,k})}$$
3. *Reduce the number of components: (Sec. III-B3)*
$$\mathrm{M}(\alpha_{j,k}, \mu_{j,k}, \mathrm{P}_{j,k})_{(j, N_{\mathrm{GMF},k})}$$

endfor

A specific GMF for BS position estimation is explained next:

1) Initialization of GMF:

The initialization is done using the first TA measurement. The idea is to compute Gaussian mixture approximation of the posterior distribution at time t_1. Because the initial prior distribution is uniform, we can compute Gaussian mixture approximation of the first likelihood function $p(\mathrm{TA}_1|\mathbf{x}^{\mathrm{BS}})$. In this case we use

$$\mathrm{M}(\alpha_{j,1}, \mu_{j,1}, \mathrm{P}_{j,1})_{(j, N_{\mathrm{GMF},1})}, \quad (14)$$

where

$$\alpha_{j,1} = \frac{1}{N_{\mathrm{GMF},1}}$$

$$\mu_{j,1} = \mathrm{Q}_j \frac{\int_A \mathbf{x}^{\mathrm{BS}} p(\mathrm{TA}_1 \mid \mathbf{x}^{\mathrm{BS}}) d\mathbf{x}^{\mathrm{BS}}}{\int_A p(\mathrm{TA}_1 \mid \mathbf{x}^{\mathrm{BS}}) d\mathbf{x}^{\mathrm{BS}}} \approx \mathrm{Q}_j \begin{bmatrix} \mu_{ta} \\ 0 \end{bmatrix}$$

$$\mathrm{P}_{j,1} = \mathrm{Q}_j \frac{\int_A (\mathbf{x}^{\mathrm{BS}} - \mu_{j,1})(\mathbf{x}^{\mathrm{BS}} - \mu_{j,1})^T p(\mathrm{TA}_1 \mid \mathbf{x}^{\mathrm{BS}}) d\mathbf{x}^{\mathrm{BS}}}{\int_A p(\mathrm{TA}_1 \mid \mathbf{x}^{\mathrm{BS}}) d\mathbf{x}^{\mathrm{BS}}} \mathrm{Q}_j^T$$

$$\mathrm{Q}_j = \begin{bmatrix} \cos\frac{2\pi j}{N_{\mathrm{GMF},1}} & \sin\frac{2\pi j}{N_{\mathrm{GMF},1}} \\ -\sin\frac{2\pi j}{N_{\mathrm{GMF},1}} & \cos\frac{2\pi j}{N_{\mathrm{GMF},1}} \end{bmatrix}$$

$$A = \left\{ \mathbf{x} \,\middle|\, \frac{\mathbf{x}_1}{\|\mathbf{x}\|} \geq \cos\left(\frac{\pi}{N_{\mathrm{GMF},1}}\right) \right\}.$$

In our implementation the number of components used is $N_{\mathrm{GMF},1}$ and $\mathrm{P}_{j,1}$ is computed numerically.

2) Update:

Here we convert likelihood to the range measurement model (for simplicity we suppress the subscript k)

$$\mu_{ta_j} = \mathrm{h}_j(\mathbf{x}^{\mathrm{BS}}) + \mathbf{v}_j = \left\|\mathbf{x}^{\mathrm{BS}} + \mathbf{x}_j^{\mathrm{M}}\right\| + \mathbf{v}_j, \quad (15)$$

where $\mathbf{v} \sim \mathrm{N}(0, \sigma_{ta_j}^2)$. Now the posterior approximation is

$$\mathrm{M}(\alpha_j^+, \mu_j^+, \mathrm{P}_j^+)_{(j, N_{\mathrm{GMF}^+})}, \quad (16)$$

where

$$\mathrm{H}_j = \begin{cases} \dfrac{(\mu_j^- - \mathbf{x}^{\mathrm{M}})^T}{\left\|\mu_j^- - \mathbf{x}^{\mathrm{M}}\right\|} & \text{,if } \left\|\mu_j^- - \mathbf{x}^{\mathrm{M}}\right\| \neq 0 \\ [0\,0] & \text{,otherwise} \end{cases}$$

$$\mathrm{K}_j = \frac{1}{\mathrm{H}_j \mathrm{P}_j^- \mathrm{H}_j^T} \mathrm{P}_j^- \mathrm{H}_j^T$$

$$\alpha_j^+ \propto \alpha_j^- e^{-\frac{(\mu_{ta_j} - \mathrm{h}_j(\mu_j^-))^2}{2(\mathrm{H}_j \mathrm{P}_j^- \mathrm{H}_j^T + \sigma_{ta_j}^2)}}$$

$$\mu_j^+ = \mu_j^- + \mathrm{K}_j(\mu_{ta_j} - \mathrm{h}_j(\mu_j^-))$$
$$\mathrm{P}_j^+ = (\mathrm{I} - \mathrm{K}_j\mathrm{H}_j)\mathrm{P}_j^- .$$

After the above computations are done for each component, the weights are normalized so that $\sum_{j=1}^{N_{\mathrm{GMF}+}} \alpha_j^+ = 1$.

3) Reduce the Number of Components:

Components are reduced using two different methods [11, 14, 15]. In the first method, components having a weight α_j^+ less than some threshold are simply dropped and the weights of the other components are renormalized. In the second method components are merged, if they are close to each other i.e. $\left\|\mu_i^+ - \mu_j^+\right\| < \theta$, where θ is the component merge distance. The parameters for the new merged component may be written as follows

$$\mu_l = \frac{\alpha_i^+ \mu_i^+ + \alpha_j^+ \mu_j^+}{\alpha_i^+ + \alpha_j^+} \tag{17}$$

$$\begin{aligned} \mathrm{P}_l = &\frac{\alpha_i^+}{\alpha_i^+ + \alpha_j^+}\left[\mathrm{P}_i^+ + (\mu_i^+ - \mu_l)(\mu_i^+ - \mu_l)^T\right] + \\ &\frac{\alpha_j^+}{\alpha_i^+ + \alpha_j^+}\left[\mathrm{P}_j^+ + (\mu_j^+ - \mu_l)(\mu_j^+ - \mu_l)^T\right] \end{aligned} \tag{18}$$

$$\alpha_l = \alpha_i^+ + \alpha_j^+ \tag{19}$$

IV. RESULTS

A. Model Fitting

In Fig. 2 the cumulative probability distributions of the measurements shown in Fig. 1 are plotted. Compared to the ideal model the TA measurements are clearly

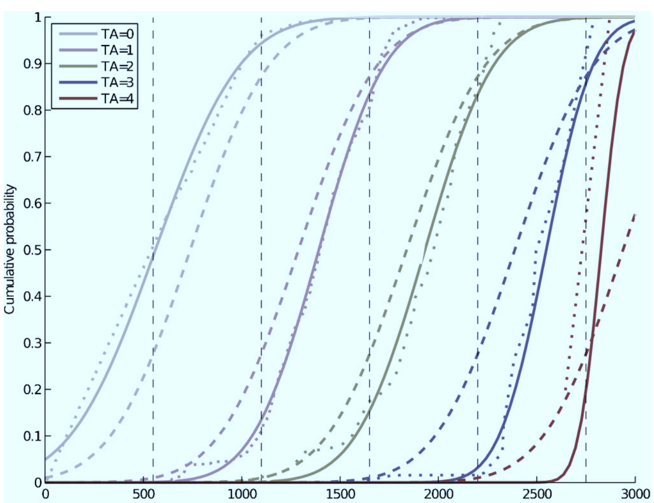

Figure 2. Fitted CDF functions. Solid line is the Gaussian fit with separate mean and variance for each TA, dashed line is a single model fit, dotted line is the original cumulative density function and vertical dashed lines present the limits of the ideal model

expanded to larger areas and have means close to upper limit of the ideal model.

B. BS Positioning

BS positioning is done using the PMF and GMF. The component merge distance is θ=100m and the component discard value is 10^{-6}.

1) Simulated Results:

In simulations 20-100 TA measurements are simulated using likelihood

$$p(\mathrm{TA} \mid \mathbf{x}^{\mathrm{BS}}) = \begin{cases} 1, 0 \le \left\|\mathbf{x}^{\mathrm{M}} - \mathbf{x}^{\mathrm{BS}}\right\| - \mathrm{TA}\Delta_{\mathrm{TA}} < \Delta_{\mathrm{TA}} \\ e^{-\frac{\left|\left\|\mathbf{x}^{\mathrm{M}} - \mathbf{x}^{\mathrm{BS}}\right\| - \frac{\mathrm{TA}\Delta_{\mathrm{TA}}}{2}\right|}{2\Delta_{\mathrm{TA}}^{ta}}}, \text{otherwise} \end{cases}, \tag{20}$$

where $\Delta_{\mathrm{TA}}^{ta} \ge \Delta_{\mathrm{TA}}$ is a width parameter that expands some TA measurements more than others. Exponential tails are also used to model measurement errors.

Measurement models for GMF are fitted using 1000 samples generated using random measurement positions and the likelihood above. The ideal model uses the mean and variance of (3). Single model is done using (8) and multiple models are computed with (6) and (7).

Table I presents the simulation results. The reference solution is computed using PMF with the optimal model and 90000 points (300×300 grid).

TABLE I. SIMULATED BS POSITIONING RESULTS

Filter	Mean error [m]	Mean reference error [m]
Ideal model GMF	442	414
Single model GMF	349	322
Multiple model GMF	160	84
PMF	141	0

2) Real BS Positioning

The real measurements were collected in a suburban area using a Nokia mobile phone that was calling all the time to get the TA measurements.

In Fig. 3 the grid filter and GMF are compared. The probability distribution of estimated BS position is shown in red. The blue triangle is the true BS position. In this case the likelihood given in (20) is used with $\Delta_{\mathrm{TA}}^{ta}$ = 550m for all TA measurement values and the GMF measurement model means were the ideal means but variance 4 times bigger. Fitted models are not used in this case to prevent filters from having any extra information of the BS position.

V. FUTURE WORK

In future the effect of base station position in user positioning should be evaluated. The effect of using TA measurement should be compared to cell model positioning without TA measurement.

More measurements from different BSs would be needed to make a good model of real TA measurements and use that

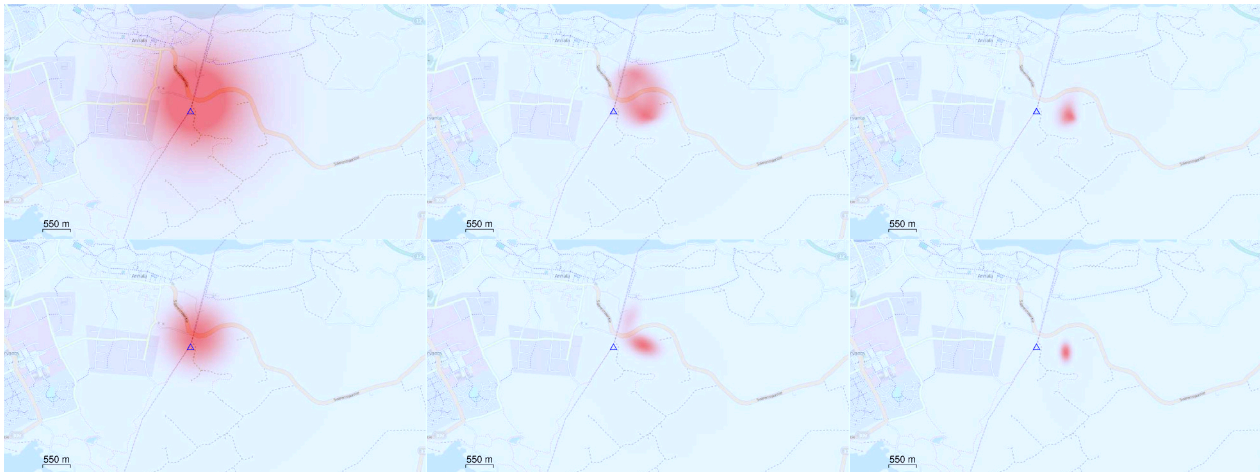

Figure 3. PMF (top) and GMF (bottom) distributions after 1, 10 and 30 real measurements

model in BS positioning case that is independent of the modeling case. Also a measurement model should cover all 63 TA values –we measured only 5 values. A measurement campaign should be done in rural environment to get measurements with higher TA values.

Other things to do in future might be to account for the fact that most of cells are not round but more like sectors and to investigate the error models further.

VI. CONCLUSIONS

This paper showed that the BS position may be solved using TA measurements from known locations. Simulated results show that the accuracy of GMF is close to ideal when the measurement errors are well modeled. On other hand the real measurement case showed that even with a crude model the BS position may be estimated with accuracy close to the granularity of TA measurements.

One problem with TA measurements is that they are not available when the radio link is in idle state. To avoid this phone may open a data connection before the positioning. Another issue is that the UMTS (Universal Mobile Telecommunications System) networks using WCDMA (Wideband Code Division Multiple Access) measure RTT, but the network does not provide RTT to mobile phone. So this feature can be used only on the network side.

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